Collocation-based harmonic balance method for highly accurate periodic solutions of the Three-body system

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Abstract: The libration point periodic orbit of the circular restricted three-body problem (CRTBP) is attracting ever-growing attention. The traditional technique of using the analytical initial value and numerical differential correction (DC) method has issues such as limited accuracy, changes in design parameters, and orbit closure. Therefore, we proposed a semi-analytical collocation-based harmonic balance framework - reconstruction harmonic balance method (RHB) to restudy this problem. The simulation results show that the method proposed can obtain closed periodic solutions with higher accuracy than the DC method. It is also easy to obtain complete families of periodic orbits and determine bifurcation points in conjunction with a compact parameter-sweeping technique.

Keywords: CRTBP, Periodic orbit families, Differential correction method, Reconstruction harmonic balance method, Parameter-sweeping

1. Introduction

The circular restricted three-body problem (CRTBP) is a special kind of three-body problem which is widely used in the field of deep space exploration orbit design\(^{[1, 2]}\). There is no analytical solution to the CRTBP, but there are five libration points and a series of periodic orbits. Among them, the periodic orbit of the three-body system can not only be used directly as the mission orbit of the spacecraft to achieve lunar back communication \(^{[3]}\), solar exploration \(^{[4]}\), and other tasks \(^{[5, 6]}\), but also can be used to construct low-energy transfer in the three-body system\(^{[7]}\). Therefore, solving the periodic orbit of the restricted three-body problem is of great significance.

Farquhar et al. \(^{[8]}\) first studied the high-order analytical expression of periodic and quasi-periodic orbits considering the influence of solar gravity. Breakwell et al. \(^{[9]}\) then studied the halo orbit in the Earth-Moon CRTBP circumstance and employed the differential correction method to obtain the halo orbit numerically. Using the same approach, Howell et al. \(^{[10]}\) studied the halo orbits near all collinear libration points for different mass parameters. However, these two studies introduce little about how to obtain an appropriate initial value to initiate the iteration procedure because convergence highly depends on the accuracy of the initial value. To this end, Richardson \(^{[11]}\) employed the classical Lindstedt-Poincare method to derive a three-order analytical expression of the halo orbit. Because of the excellent accuracy, the three-order result combined with the differential correction procedure was deemed a common-used method to calculate halo orbits. Same using the Lindstedt-Poincare method and combining the Fourier series, Gomez et al. \(^{[12]}\) also obtained the high-order expansion of the halo orbit. But like
methods in ref. [8] and [11], the accuracy of periodic orbits tend to deteriorate as the orbits’ size increase making the orbits fail to be used in practical missions. Although the differential correction method used in ref. [9] and [10] permits more accurate solutions. It has to solve a system with 42 differential equations, making the computation rather time-consuming. Moreover, the method may change the prescribed parameter, and the final orbits obtained are not strictly closed but within a certain error tolerance.

Considering the above problems, we start from a new perspective using the time domain collocation (TDC) method to restudy the problem of solving the periodic orbit in CRTBP. The TDC method proposed by Dai et al. [13] is essentially one of the practical residual methods and has been successfully used to solve various nonlinear dynamical problems [14-16]. Wang et al. [17] also introduced this method into astrodynamics to search for the periodic orbits of satellite relative motion precisely. In this paper, based on the ideas of Dai et al. [13], we propose a semi-analytical collocation-based harmonic balance framework-reconstruction harmonic balance method (RHB) to solve the periodic orbits in the CRTBP. Methods to construct RHB algebraic system are first introduced. Considering that non-polynomial nonlinearity contained in the equation would inherently introduce aliasing errors, we employ a recasting strategy to improve the calculation accuracy. We also study the periodic orbit families in CRTBP. The initial values of the algebraic equations can be provided by the approximate analytical method or random Monte Carlo simulation method. Then, the complete periodic orbit families can be obtained by using the compact parameter-sweeping approach. The characteristics of the periodic orbit families of both $L_1$ and $L_2$ points are analyzed.

2. Fundamentals of constructing the RHB algebraic equation

To implement the RHB method, the motion components of the three-dimensional space are firstly expressed in the form of a truncated Fourier expansion as

$$f = f_0 + \sum_{n=1}^{N} f_{2n-1} \cos(n\omega t) + f_{2n} \sin(n\omega t). \quad (1)$$

Here $f$ is the periodic approximate solution of equation to be solved, $N$ is the number of harmonics included in the approximation, $\omega$ is the supposed frequency of the periodic motion and $f_i$ ($i=1, 2, \ldots, 2n$) are the unknown Fourier coefficient variables. Obviously, the results of RHB method will be inherently periodic.

Collocating $f(t_i)$ ($i = 1, 2, \ldots, M$) at $M$ time points equally spaced in a period $T$ of $f(t)$, then from (1), we get

$$f \left( t_i \right) = f_0 + \sum_{n=1}^{N} f_{2n-1} \cos(n\omega t_i) + f_{2n} \sin(n\omega t_i), \quad (2)$$

Where, $t_i = 2\pi (i-1)/(\omega M)$ ($i = 1, 2, \ldots, M$). Defining $\tilde{f} = [f(t_1), \ldots, f(t_M)]^T$, $\hat{f} = [f_0, \ldots, f_{2n}]^T$. Then (2) can be simplified into

$$\tilde{f} = E\hat{f}, \quad (3)$$

Where, $E$ is the transformation matrix
\[
\mathbf{E} = \begin{bmatrix}
1 & \cos(\omega t_1) & \sin(\omega t_1) & \cdots & \cos(N\omega t_1) & \sin(N\omega t_1) \\
1 & \cos(\omega t_2) & \sin(\omega t_2) & \cdots & \cos(N\omega t_2) & \sin(N\omega t_2) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \cos(\omega t_M) & \sin(\omega t_M) & \cdots & \cos(N\omega t_M) & \sin(N\omega t_M)
\end{bmatrix}_{M \times (2N+1)}.
\]  

Thus, if we obtain the value of \(M\) time points \(f(t_i)\), the Fourier coefficient can be determined by \(\hat{f} = \mathbf{E}^+ \mathbf{f}\), where \(\mathbf{E}^+\) is the pseudoinverse matric of \(\mathbf{E}\) with the explicit expression being:

\[
\mathbf{E}^+ = \frac{2}{M} \begin{bmatrix}
1 & 1 & \cdots & 1 \\
\frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\
\cos(\omega t_1) & \cos(\omega t_2) & \cdots & \cos(\omega t_M) \\
\sin(\omega t_1) & \sin(\omega t_2) & \cdots & \sin(\omega t_M) \\
\cos(2\omega t_1) & \cos(2\omega t_2) & \cdots & \cos(2\omega t_M) \\
\sin(2\omega t_1) & \sin(2\omega t_2) & \cdots & \sin(2\omega t_M) \\
\vdots & \vdots & \ddots & \vdots \\
\cos(N\omega t_1) & \cos(N\omega t_2) & \cdots & \cos(N\omega t_M) \\
\sin(N\omega t_1) & \sin(N\omega t_2) & \cdots & \sin(N\omega t_M)
\end{bmatrix}
\]  

Using expression (2), the first order time derivative of \(f(t_i)\) can be written as

\[
\dot{f}(t_i) = \frac{d}{dt} f(t_i) = \sum_{n=1}^{N} n\omega f_{2n-1} \cos(n\omega t_i) + n\omega f_{2n} \cos(n\omega t_i).
\]  

Referring to derivation of (3), the relationship between Fourier coefficient and the time derivatives \(\dot{f}(t_i)\) can be expressed in a subtle form

\[
\hat{\dot{f}} = \omega \mathbf{E} \hat{f},
\]  

Where

\[
\mathbf{A} = \begin{bmatrix}
0 & \mathbf{J}_1 & \cdots & \mathbf{J}_n \\
\mathbf{J}_1 & \mathbf{J}_2 & \cdots & \mathbf{J}_n \\
\mathbf{J}_2 & \mathbf{J}_3 & \cdots & \mathbf{J}_n \\
\vdots & \vdots & \ddots & \vdots \\
\mathbf{J}_n & \mathbf{J}_n & \cdots & \mathbf{J}_n
\end{bmatrix}_{(2N+1) \times (2N+1)}, \quad \mathbf{J}_n = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
\]

Continue to take the second order time derivative of \(f(t_i)\), we can obtain

\[
\ddot{f} = \omega^2 \mathbf{A}^2 \hat{f} = \omega^2 \mathbf{E} \mathbf{A}^2 \mathbf{E}^+ \hat{f}.
\]  

Thus, the transformation from \(f(t_i)\) to \(\ddot{f}(t_i)\) is established.
In the following parts, we will use the preceding equations and transformations to derive the RHB algebraic system for the nonlinear circular restricted three-body problem, thereby approximating the periodic solution in the vicinity of libration points.

Since the motion near the collinear libration point is of interest, it is convenient to study the motion relative to the collinear libration point. In this new coordinate system, the origin of coordinate is located in the collinear libration point $L_j$ and the direction of $x$ axis and $y$ axis are the same as the synodic coordinate frame [18]. Meanwhile, in order to make the moving image near the libration point clear, a small scale factor $\gamma$ is introduced the enlarge the distance scale. Thus, the equation of motion is

\[
\begin{cases}
\dot{x} - 2\dot{y} = -\frac{1}{\gamma^2} \bar{U}_x, \\
\dot{y} + 2\dot{x} = -\frac{1}{\gamma^2} \bar{U}_y, \\
\ddot{z} = -\frac{1}{\gamma^2} \bar{U}_z,
\end{cases}
\]

(9)

Where

\[
\bar{U}(x, y, z) = -\frac{1}{2}[(\gamma_i x + X_i)^2 + (\gamma_i y)^2] - \frac{1-\mu}{r_1} - \frac{\mu}{r_2} - \frac{1}{2} \mu(1-\mu),
\]

\[
r_1 = \sqrt{(\gamma_i x + X_i + \mu)^2 + (\gamma_i y)^2 + (\gamma_i z)^2},
\]

\[
r_2 = \sqrt{(\gamma_i x + X_i + \mu - 1)^2 + (\gamma_i y)^2 + (\gamma_i z)^2},
\]

and $\bar{U}_x$, $\bar{U}_y$ and $\bar{U}_z$ are the partial derivatives of $\bar{U}$ in $x$, $y$ and $z$ direction respectively. Apparently, the equation of motion near the libration point is a non-polynomial type nonlinear system which cannot be efficiently solved by existing method. In the process of solving, another recasting method is introduced to make the equation more easily solved by the proposed RHB method. For more detail, one can see appendix of this paper.

For simplicity, we take the $\bar{X} = f(X, \dot{X}, t)$ on behalf of (9). Collocating $M$ points in a time period $T$ of $\bar{X} = f(X, \dot{X}, t)$, then we get

\[
\begin{pmatrix}
\bar{X}(t_1), \bar{X}(t_1), \ldots, \bar{X}(t_M)
\end{pmatrix}^T = \begin{pmatrix}
f(X_1, \dot{X}_1, t_1), f(X_2, \dot{X}_2, t_2), \ldots, f(X_M, \dot{X}_M, t_M)
\end{pmatrix}^T.
\]

(10)

Recalling (8), the left hand of (10) can be converted to

\[
\begin{bmatrix}
\omega^2 E_x A E_x^+ \\
\omega^2 E_y A E_y^+ \\
\omega^2 E_z A E_z^+
\end{bmatrix}
\begin{bmatrix}
\bar{x} \\
\bar{y} \\
\bar{z}
\end{bmatrix},
\]

(11)
With the order of component in (10) being rearranged. Here, \( \tilde{X} = [\tilde{x}, \tilde{y}, \tilde{z}]^T \) and \( \tilde{x}, \tilde{y} \) and \( \tilde{z} \) represent the value of components in the three-dimensional space at each collocation point, respectively. For instance, \( \tilde{x} = [x(t_1), x(t_2), ..., x(t_M)]^T \). Besides, \( E_x, E_y \) and \( E_z \) are transformation matrixes corresponding to \( \omega_x, \omega_y \) and \( \omega_z \) separately. For brevity, define

\[
\tilde{E} = \begin{bmatrix}
E_x & E_y & E_z
\end{bmatrix}, \quad \tilde{A} = \begin{bmatrix}
A & A & A
\end{bmatrix}, \quad \tilde{E}^+ = \begin{bmatrix}
E_x^+ & E_y^+ & E_z^+
\end{bmatrix},
\]

then (11) can be further simplified into

\[
\dot{\tilde{X}} = \omega^2 \tilde{E} \tilde{A} \tilde{E}^+ \tilde{X} = \tilde{f}(\tilde{X}, \omega \tilde{E} \tilde{A} \tilde{X}, t).
\]

According to the relationship \( \check{f} = E^+ \tilde{f} \), we get the RHB algebraic system of (9):

\[
\omega^2 \tilde{E} \tilde{A} \hat{X} = \tilde{f}(\tilde{X}, \omega \tilde{E} \tilde{A} \hat{X}, t),
\]

where \( \hat{X} = [x_0, ..., z_{2n}]^T \), is the unknown variables of the equation. \( \tilde{f}(\tilde{X}, \omega \tilde{E} \tilde{A} \hat{X}, t) \) is the rearranged version of the vector \( \{f(X_1, \hat{X}_1, t_1), f(X_2, \hat{X}_2, t_2), ..., f(X_M, \hat{X}_M, t_M)\}^T \).

To solve this algebraic system, the Newton method can be very appropriate [17]. We derive the Jacobian matrix \( J \) in a numerical way. Denote \( R = \omega^2 \tilde{E} \tilde{A} \hat{X} - \tilde{f}(\tilde{X}, \omega \tilde{E} \tilde{A} \hat{X}, t) \), and assume that \( \hat{X} = \tilde{X} + (0, ..., \delta \hat{X}_i, ..., 0)^T \), with \( \delta \hat{X}_i \) being a small displacement from \( \hat{X}_i \), which is the \( i \)th column of \( \hat{X} \). Then the Jacobian matrix can be computed by \( J_i(\hat{X}) = (R(\hat{X}) - R(\tilde{X}))/\delta \hat{X}_i \). Thus, once the initial value of \( \hat{X} \) is obtained, then \( \hat{X} \) can be determined by \( \hat{X} = E^+ \tilde{X} \) and be used to initialize the iteration algorithm.

3. Strategy for solving periodic orbit family in CRTBP

In the previous section, we introduced the basic theory for constructing the RHB algebraic equation. After solving this equation, the coefficients of the Fourier series expansion can be obtained, and thus the motion state of the CRTBP system can be determined. However, a potential problem is that by directly using the original equation of CRTBP, the periodic solution's accuracy would not be very high. That is because the original equation contains non-polynomial terms, so we use the recasting strategy here to improve the solution accuracy. In addition, since we solve the constructed algebraic equations using Newton's iterative method, the actual system has multiple solutions. The initial value will affect not only the convergence of the solution but also the type of the obtained orbit. Therefore, the method of producing the initial values is introduced here. Finally, a compact parametric-sweeping method is employed to obtain the intact families of periodic orbits.

3.1 Recasting process

Apparently, equation (9) involves non-polynomial nonlinearity, although it still can be solved by the proposed RHB method directly, they will produce inevitable aliasing errors [19]. Herein, we
propose to use a simple recasting technique that can equivalently convert the CRTBP into polynomial types, so that the RHB method can tackle the problem without aliasing. Specifically, for recasting purpose, we need to introduce new variables

\[ u = r_1^{-3}, \quad v = r_2^{-3}. \]  

Thus, equation (9) can be equivalently converted into the following polynomial type differential-algebraic system:

\[
\begin{align*}
\dot{x} - 2\dot{y} &= -1/\gamma[-(\gamma x + X_i) + (1 - \mu)(\gamma x + X_i + \mu)u + \mu(\gamma x + X_i - 1 + \mu)v], \\
\dot{y} + 2\dot{x} &= y - (1 - \mu)y u - \mu y v, \\
\dot{z} &= -(1 - \mu)zu - \mu z v, \\
0 &= ur_1^3 - 1, \\
0 &= vr_2^3 - 1, \\
0 &= (\gamma x + X_i + \mu)^2 + (\gamma y)^2 + (\gamma z)^2 - r_1^2, \\
0 &= (\gamma x + X_i + \mu - 1)^2 + (\gamma y)^2 + (\gamma z)^2 - r_2^2.
\end{align*}
\]

This transformation supplies the RHB method with polynomial nonlinear equations which can effectively remedy the aliasing error caused by non-polynomial system. In computations, the state of the calculation results at a certain time is used to provide the initial value.

3.2 Initial value determination

For the solution of the periodic orbit in CRTBP, the selection of the initial value is important. The initial value can be given by a low-order analytical solution. The first-order expansion of Equation (9) is carried out, and the approximate analytical form of the periodic motion near the libration point can be expressed as

\[
\begin{align*}
x &= -A_x \cos(\omega_x t + \phi), \\
y &= \kappa A_x \sin(\omega_x t + \phi), \\
z &= A_z \sin(\omega_z t + \psi),
\end{align*}
\]

where, \( A_x \) and \( A_z \) are the amplitudes of motion in the x and z directions, respectively. And \( \omega_x \) and \( \omega_z \) represent the corresponding frequency of motion. Equation (17) can easily produce initial values for the planar Lyapunov orbital or the vertical Lyapunov orbital. For halo orbit, the Richardson third-order approximation solution [11] is recommended.

In addition to the above analytical methods, the Monte Carlo method can also be used to set the initial value randomly. That is to say, we randomly give thousands of groups of initial values to study how many groups of solutions exist for the RHB equations at a particular frequency. In the actual solution process, it is found that the first few harmonic coefficients are often more prominent in value. Therefore, in solving the initial value, only the first few harmonic coefficients can be considered and randomly selected in a specific range, for example, \([-1, 1]\).

3.3 Parameter-sweeping procedure

To obtain a periodic orbit, it is necessary to ensure that \( \omega_x = \omega_y = \omega_z \neq \omega \) (\( \omega_x = \omega_y = \omega_z \) for two-dimension) in equation(11), that is, there is only one fundamental orbital frequency in the system.
Therefore, it is possible to use the simple parameter sweep method for solving a periodic orbital family in a wide frequency range, regarding the orbital frequency as a continuous variable. Specifically, for a particular frequency $\omega_i$, the solution of the previous orbit corresponding to $\omega_{i-1}$ can be used as the initial value. However, it is worth noting that although this simple continuation method is relatively easy to implement in programming, when a bifurcation point is encountered, the calculation speed will slow down, and only a single branch can be obtained. A practical method is to perform the above-mentioned Monte Carlo shooting to re-obtain the initial value solution at this frequency and then perform the parameter-sweeping for each solution.

4. Numerical Results and Analysis

In the following part, the numerical simulations demonstrate the effectiveness of the RHB approach in obtaining the periodic orbit families in the CRTBP. Targeting at the Earth-Moon system whose mass parameter $\mu = 0.01215$, computation results for the halo orbits are shown in Fig.1. Using the RHB method, we take the halo orbit of the Queqiao relay satellite as an example. Concretely, the RHB method is employed to produce a highly accurate nominal orbit whose accuracy is verified by numerical propagation. Figure 1 shows the trajectories that numerically propagate the initial states from RHB with different orders. To evaluate the station-keeping ability, we set a small section on the $xOz$ plane and record the number of times the orbit pass through the section before it deviates. From Fig.2, it is seen that the DC method can provide a reference trajectory drifting away after $4.5T$ ($T$ is the orbital period) at its highest accuracy. And it appears that the higher order RHB method leads to a better orbit-keeping property, leaving more intersection points on the section. Moreover, the RHB50 solution outperforms the best DC solution, indicating that the RHB method can obtain a more accurate periodic solution.

![Figure 1. Halo orbits obtained by the RHB method with different harmonics](image-url)
Figure 2. The change of the distance between different orbits obtained and the $L_2$ point

To evaluate the effect of the recasting process, Fig.3a-3d show the error of RHB results against a high-fidelity benchmark result obtained by RHB50. It is seen that under the same convergence criterion, the computational accuracy of the RHB with the recasting technique is apparently better than that without recasting.

Combined with the parameter-sweeping approach, we use the continuation method to generate the periodic family by extending the orbital frequency. The range of orbital frequency is prescribed as [1, 3.5]. The approximate solution obtained by the simplified Richardson model or Monte Carlo method can be used to supply the RHB process with initials. Three typical periodic orbit families in the vicinity of two collinear points, $L_1$ and $L_2$, corresponding to the planar Lyapunov orbit family, the vertical Lyapunov orbit family, and the halo orbit family are shown in Fig. 4 and Fig. 5, respectively. Apart from the libration point orbits family, a kind of stable periodic orbit named as distant retrograde orbit family shown in Fig. 6 is also easily obtained. Actually, more periodic families in the three-body system, local or global, can be captured with the present method if their initial valued are determined.

Figure 3. Computational errors for solving original and recast systems with different RHB orders. (a) RHB10. (b) RHB15. (c) RHB20. (d) RHB25.

Moreover, the amplitude-frequency response curves of the above periodic orbit families are shown in Fig.7. Next, we will briefly analyze the characteristics of the obtained periodic orbit
families to help readers grasp the fundamental evolution law of different orbit families with orbital frequency. Planar Lyapunov orbit families of both $L_1$ and $L_2$ points show a similar trend. As the frequency increases, the orbits shrink in amplitude and approach the libration points. Although the vertical Lyapunov orbit families of $L_1$ and $L_2$ points show decreasing amplitude with increasing frequency in the $z$ direction, the trends and magnitudes of the amplitudes in the $x$ and $y$ directions are significantly different. Interestingly, due to the influence of gravity, the vertical Lyapunov of $L_1$ primarily opens toward the Earth at a low frequency and gradually bends toward the moon as the frequency increases. At the same time, the $L_2$ family always faces the direction of the moon. The halo orbit families act as the bifurcation of the planar Lyapunov orbit families with a beginning frequency of about $\omega = 2.26$ of $L_1$ point and about $\omega = 1.84$ of $L_2$ point, separately. Among them, the $L_1$ family is slightly more complicated because a frequency may correspond to two sets of amplitude branches. From the shape perspective, as the frequency increases, the orbit gradually drifts from the libration point to the moon along the first branch, and the two branches merge and terminate around $\omega = 3.32$. After that, if the frequency is swept backward along the second branch, the orbital shape will gradually stretch from an ellipse to a droplet one and reach a larger amplitude, terminating at about $\omega = 2.02$. In contrast, the evolution of the $L_2$ point halo family is more apparent. The orbit family begins near the bifurcation point of the Planar Lyapunov orbit and eventually approaches a near-rectilinear halo orbit close to the moon.

![Figure 4. Three types of classic periodic orbit in the vicinity of Earth-Moon $L_1$ point. a, Planar Lyapunov orbit family. b, Vertical Lyapunov orbit family. C, Halo orbit family.](image)
Figure 5. Three types of classic periodic orbit in the vicinity of Earth-Moon $L_2$ point. a, Planar Lyapunov orbit family. b, Vertical Lyapunov orbit family. C, Halo orbit family.

Figure 6. Distant retrograde orbit family in the Earth-Moon system.
Figure 7. Amplitude-frequency response curves of periodic orbit in the Earth-Moon system. a, $L_1$ point. b, $L_2$ point.

5. Conclusion

In this paper, we propose a semi-analytical collocation-based harmonic balance framework-reconstruction harmonic balance method (RHB) to solve the periodic orbits in the CRTBP. By assuming that the periodic orbits are expressed by truncated Fourier expansions and doing some transformations, the RHB algebraic system for the restricted three-body system is constructed with respect to the Fourier series coefficients. Using the Newton-Raphson method to solve this system, we can obtain the position and velocity of each collocation point within a period. Moreover, considering that non-polynomial nonlinearity contained in the equation would inherently introduce aliasing errors, we employ a recasting strategy to improve the calculation accuracy.

We also study the periodic orbit families in the three-body system. The initial values of the algebraic equations are randomly given by the Monte Carlo simulation method, and the orbital frequency is used as the continuation variable. Then, the complete periodic orbit family can be obtained by the compact parameter-sweeping approach. The characteristics of the periodic orbit families of both $L_1$ and $L_2$ points are analyzed. Specifically, the following conclusions were drawn:

- As the number of harmonics increases, the accuracy of the RHB method will be improved. Combined with the recasting technique, the higher-order RHB method can achieve a more accurate and closed periodic orbit than the traditional differential correction approach.

- More initial values of periodic orbit could be obtained by using the Monte Carlo method. Taking the orbital frequency as the continuation variable, the complete periodic orbital family can be easily obtained by the compact parameter sweep approach, and the bifurcation points of the orbit family can also be revealed.

Although the present method is very efficient and convenient for calculating the periodic orbit families in the three-body system, some problems are also found in its practical use. One problem is that if a particular orbit type has a bifurcation point at a specific frequency, the continuation speed of this point will be slowed down. A more advanced continuation method will be employed later to improve the continuation efficiency, and the bifurcated branches can be
directly obtained. In addition, the calculated orbit accuracy may be degraded to a certain extent with the continuation process, and these problems will be solved and revealed in the follow-up research. Moreover, the present research can be further studied in the following directions. First, the method can be extended into the elliptical restricted three-body model, four-body model, and even ephemeris model to obtain high-precision periodic orbit families. Secondly, this paper only studies the solution of periodic orbits in CRTBP. Still, this system has a series of quasi-periodic orbits, such as Lissajous or Quasi-halo, which also have broad application prospects. Hence, the method proposed in this paper can be further improved and used to explore the quasi-periodic orbit solution problems. The above issues will also be revealed in the follow-up research.

6. References


